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Towers of surfaces dominated by products of curves and elliptic curves of large rank over function fields ☆

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Abstract

With the goal of producing elliptic curves and higher-dimensional abelian varieties of large rank over function fields, we provide a geometric construction of towers of surfaces dominated by products of curves; in the case where the surface is defined over a finite field our construction yields families of smooth, projective curves whose Jacobians satisfy the conjecture of Birch and Swinnerton-Dyer. As an immediate application of our work we employ known results on analytic ranks of abelian varieties defined in towers of function field extensions, producing a one-parameter family of elliptic curves over $\mathbb{F}_q(t^{1/d})$ whose members obtain arbitrarily large rank as $d \to \infty$.

Keywords: Elliptic curves; L-functions; Birch and Swinnerton-Dyer conjecture; Abelian varieties

1. Introduction

1.1. Let E/K denote an elliptic curve defined over a global field K. By a theorem of Mordell and Weil the group $E(K)$ of K-points on E is a finitely generated abelian group, and so has the structure:

$$E(K) \simeq E_{\mathrm{tors}}(K) \times \mathbb{Z}^r,$$

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where r denotes the rank. It is conjectured, for any fixed global field K, that there exist elliptic curves with Mordell–Weil groups $E(K)$ of arbitrarily large rank; the largest known rank for $K = \mathbb{Q}$ at this writing is 28, due to Elkies. In the case where $K = \mathbb{F}_q(t)$ the conjecture is a theorem, but a limited class of examples exist. This work is due to Shafarevich and Tate, and Ulmer [Ulm02]. The curves of Shafarevich and Tate are *isotrivial*: their j-invariants lie in the field of constants. Ulmer shows that the non-isotrivial curve with affine model $y^2 - xy = x^3 + t^d$ over $\mathbb{F}_q(t)$ obtains arbitrarily large rank as $d \to \infty$.

1.2. The L-series $L(E/K, s)$ of an elliptic curve, defined as an Artin L-function from the ℓ-adic representation of E/K, is an analytic invariant that encodes the reduction behavior of E/K at each place. Based on substantial numerical evidence, Birch and Swinnerton-Dyer posit the following conjectural local–global principal.

Conjecture *(Birch and Swinnerton-Dyer).*

$$\operatorname{ord}_{s=1} L(E/K, s) = \operatorname{rank} E(K).$$

We call $\operatorname{ord}_{s=1} L(E/K, s)$ the analytic rank of E/K and write BSD for the conjecture.

1.3. In Section 2 we prove a theorem that allows us to construct infinitely many towers of surfaces, each of which admits a dominant rational map from a product of curves. When such a surface is defined over a finite field, this is enough to prove BSD for the Jacobian of its generic fiber. In Section 3 we determine sufficient conditions for geometric irreducibility of the generic fibers of these surfaces, and we compute the genera of their smooth projective models. In Section 4 we explore the simplest case of our construction; we combine known results on analytic ranks of elliptic curves defined over function fields and produce a parameterized family of elliptic curves of large rank over $\mathbb{F}_q(t^{1/d})$.

1.4. This paper, which extends the work of [Ulm02], is a condensed version of the author's PhD thesis [Ber07]. The motivation for our construction comes from Shioda–Katsura, who construct Fermat surfaces dominated by products of curves. It is a pleasure to thank Bill McCallum and Dave Savitt for their useful comments, and Doug Ulmer for the question and his guidance.

2. Surfaces dominated by products of curves

2.1.

Definition 2.1. An algebraic variety V is dominated by a product of curves, *DPC*, if it admits a dominant rational map:

$$C_1 \times C_2 \times \cdots \times C_n \dashrightarrow V,$$

where the C_i denote smooth algebraic curves.

In this paper we study the particular case where the variety is a surface, S, with a dominant rational map $C_1 \times C_2 \dashrightarrow S$.

Let $\mathcal{S} \to \mathbb{P}^1$ denote a smooth fibered surface. We denote by $\mathcal{S}_d$ the surface corresponding to the base extension:

$$\begin{array}{ccc} \mathcal{S}_d & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{t \mapsto t^d} & \mathbb{P}^1. \end{array}$$

Definition 2.2. The surface $\mathcal{S}$ is dominated by a product in towers, *DPCT*, if:

(1) The surface $\mathcal{S}$ is DPC.
(2) The surfaces $\mathcal{S}_d$, defined as above, are DPC for d prime to $\mathrm{char}(k)$.

Let X_k denote a non-singular projective variety over a field k, and let $X_{\bar{k}} := X_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$. When $k = \mathbb{F}_q$, q a prime power, the zeta function of X_k is

$$\zeta(X,s) = \exp\left(\sum_{n=0}^{\infty} \frac{\#X(\mathbb{F}_{q^n})q^{-sn}}{n}\right).$$

The Néron–Severi group of X_k is the group of divisors on X_k modulo algebraic equivalence on $X_{\bar{k}}$. That the group is a finitely generated abelian group is due to Néron and Severi, and it is a conjecture of Tate [Tat65a] that:

Conjecture.

$$\operatorname{rank} NS(X) = -\operatorname{ord}_{s=1} \zeta(X,s).$$

2.2. In the case where $X \to C$ is a surface fibered over a curve over a finite field, it is a theorem that the veracity of this conjecture for X is equivalent to the conjecture of Birch and Swinnerton-Dyer for the Jacobian of its generic fiber [Tat65b]. Let V_1 and V_2 denote two varieties for which the Tate conjecture holds. It is a result of Tate [Tat94] that the conjecture is also true for the product variety $V_1 \times V_2$. Further, suppose V is any variety for which the conjecture is true, and let $V \dashrightarrow V'$ denote a dominant rational map. Then the conjecture holds for V' [Tat94].

These results imply that the Tate conjecture is true for DPC surfaces. Indeed, since algebraically equivalent divisors on a curve are those with the same degree, the rank of the Néron–Severi group of any curve is one; the Tate conjecture is true for curves, and hence for the dominated surface.

2.3. Shioda–Katsura [SK79] dominate any degree d Fermat variety F_d by a product of lower-dimensional Fermat varieties, and hence prove the Tate conjecture for any Fermat surface. Let F_d^r and F_d^s denote Fermat varieties of dimensions r and s. A dominant rational map $F_d^r \times F_d^s \dashrightarrow F_d^{r+s}$ is realized by Shioda–Katsura as a quotient of a blow-up of the product along a subvariety, and Ulmer [Ulm02] applies this in the case where $r = s = 1$, so $F_d := F_d^2$ is a surface. He constructs an elliptic surface $\mathcal{E}_d \to \mathbb{P}^1$ over $\mathbb{F}_q$, birational to a quotient of F_d, and thus proves BSD for its generic fiber, an elliptic curve over $\mathbb{F}_q(t)$.

Let h denote the rational function $\frac{x_2^d}{y_2^d}$ on the product of Fermat curves $F_d^1 \times F_d^1$. Then the rational map $h: F_d^1 \times F_d^1 \dashrightarrow \mathbb{P}^1$ is a morphism away from $x_2 = y_2 = 0$. By resolving h to a morphism one may construct the birational map of Shioda–Katsura. In what follows we use this approach, producing infinitely many DPCT surfaces, and provide a means of analyzing the ranks of the Jacobians of their generic fibers.

2.4. Schoen [Sch96] proves that the DPC property naturally extends in étale covers, and we generalize his result below:

Theorem 2.1. *Let $\varphi: C_1 \times C_2 \to \mathcal{W}$ denote a dominant morphism from a product of smooth curves to a non-singular, irreducible surface $\mathcal{W}$. Let $\rho: \mathcal{W}_d \to \mathcal{W}$ denote a degree d étale covering of $\mathcal{W}$, so that ρ is both finite and étale. Then there exist smooth, possibly open, curves $\tilde{C}_1$, $\tilde{C}_2$ and a dominant morphism*

$$\tilde{C}_1 \times \tilde{C}_2 \to \mathcal{W}_d.$$

Note that neither the curves C_1 and C_2 nor the surface $\mathcal{W}$ in the statement of the theorem are assumed to be projective.

Proof. We consider the map on fundamental groups induced by the restriction of the morphism φ to the first factor:

$$\varphi_*: \pi_1(C_1) \to \pi_1(\mathcal{W}).$$

The image $\rho_*(\pi_1(\mathcal{W}_d))$ is a finite index subgroup of $\pi_1(\mathcal{W})$, corresponding to the finite étale cover $\rho: \mathcal{W}_d \to \mathcal{W}$. Its inverse image, $\varphi_*^{-1}(\rho_*(\pi_1(\mathcal{W}_d)))$, is a subgroup of finite index in $\pi_1(C_1)$. Let $\tilde{C}_1$ denote the corresponding connected cover of C_1.

An analogous construction for $\tilde{C}_2 \to C_2$ produces a cover

$$\tilde{\rho}: \tilde{C}_1 \times \tilde{C}_2 \to C_1 \times C_2.$$

By construction, the image of $\pi_1(\tilde{C}_1 \times \tilde{C}_2)$ under the composed map of fundamental groups $\varphi_* \circ \tilde{\rho}_*$ is contained in $\rho_*(\pi_1(\mathcal{W}_d))$. Then $\tilde{\rho}': \tilde{C}_1 \times \tilde{C}_2 \to \mathcal{W}_d$ is the unique lift of $\varphi \circ \tilde{\rho}$, our desired morphism. That the morphism is dominant follows from the commutativity of the diagram below and the fact that $\rho: \mathcal{W}_d \to \mathcal{W}$ is an étale cover.

$$\begin{tikzcd}
\tilde{C}_1 \times \tilde{C}_2 \arrow[r, "\tilde{\rho}'"] \arrow[d, "\tilde{\rho}"'] \arrow[dr] & \mathcal{W}_d \arrow[d, "\rho"] \\
C_1 \times C_1 \arrow[r, "\varphi"'] & \mathcal{W}.
\end{tikzcd}$$ $\square$

2.5. Let f and g denote rational functions on a surface $\mathcal{X}$, and denote by $\frac{f}{g}: \mathcal{X} \dashrightarrow \mathbb{P}^1$ the rational map $P \mapsto (f(P), g(P))$, which is defined away from the locus of points satisfying $f = g = 0$ or $f = g = \infty$. A blow-up of $\mathcal{X}$ along this indeterminacy locus resolves the rational map to a morphism from $\mathcal{S}'$, the surface in $\mathcal{X} \times \mathbb{P}^1$ defined by the vanishing of $tf - g$. In

Section 3 we analyze the singularities of the generic fiber of a surface constructed in this manner, and we use this defining equation to determine sufficient conditions for geometric irreducibility. Here, letting $\mathcal{S}$ denote the smooth, proper, relatively minimal model of $\mathcal{S}' \to \mathbb{P}^1$, an explicit construction proves the following:

Theorem 2.2. *Let $\mathcal{C}$ and $\mathcal{D}$ denote smooth, projective curves over a field k. Let $f \in k(\mathcal{C})$ and $g \in k(\mathcal{D})$ denote separable rational functions on $\mathcal{C}$ and $\mathcal{D}$, respectively, and $\frac{f}{g}:\mathcal{C} \times \mathcal{D} \dashrightarrow \mathbb{P}^1$ the rational map defined by f/g. Let $\mathcal{S}'$ denote the surface described above, obtained via a resolution of the indeterminacy locus of $\frac{f}{g}$, and let $\mathcal{S}$ be as above. Then $\mathcal{S}$ is DPCT.*

Proof. That $\mathcal{S}$ is DPC is obvious; it is birational to $\mathcal{C} \times \mathcal{D}$. The essence of our construction of towers of DPC surfaces, and the proof of our theorem, lies in the choice of functions f, $g \in k(\mathcal{C} \times \mathcal{D})$ defining the rational map to the projective line. By choosing f from $k(\mathcal{C})$ and g from $k(\mathcal{D})$ we extend the DPC property to towers of field extensions.

Construct $\mathcal{S}'$, $\mathcal{S}$, and $\mathcal{S}_d$ as in the statement of the theorem; denote by $\frac{f}{g}$ the morphism $\mathcal{S} \to \mathbb{P}^1$ and by $(\frac{f}{g})_d$ the base change of the morphism. The surface $\mathcal{S}'$ is defined in $\mathcal{C} \times \mathcal{D} \times \mathbb{P}^1$ by the vanishing of $tf - g$, and is a birational model of $\mathcal{C} \times \mathcal{D}$. The smooth model $\mathcal{S}$ is DPC, and we show that the same holds for $\mathcal{S}_d$.

$$
\begin{tikzcd}
 & & & \mathcal{S}_d \arrow[dl, "\rho"'] \arrow[ddl, "(\frac{f}{g})_d"] \\
 & & \mathcal{S} \arrow[dll, dashed, "\beta"'] \arrow[dl, dashed] \arrow[dddll, "\frac{f}{g}"'] & \\
\mathcal{C} \times \mathcal{D} \arrow[dd, dashed, "\frac{f}{g}"'] & \mathcal{S}' \arrow[l] \arrow[ddl, "\frac{f}{g}"'] & & \\
 & & \mathbb{P}^1 \arrow[dll] & \\
\mathbb{P}^1 & & &
\end{tikzcd}
$$

The cover $\rho:\mathcal{S}_d \to \mathcal{S}$ is ramified over $t = 0$ and $t = \infty$ in $\mathbb{P}^1$. To apply Theorem 2.1 we restrict to the open subsets on which ρ is étale. Let $\mathcal{W} \subset \mathcal{S}$ denote the open subset with $V := (\frac{f}{g})^{-1}(0) \cup (\frac{f}{g})^{-1}(\infty)$ removed, and let $\mathcal{W}_d$ denote the open subset of $\mathcal{S}_d$ with $(\frac{f}{g})_d^{-1}(0)$ and $(\frac{f}{g})_d^{-1}(\infty)$ removed.

Let $\beta:\mathcal{S} \dashrightarrow \mathcal{C} \times \mathcal{D}$ denote the composition that results in the resolution of $\frac{f}{g}$ and of the surface. The inverse image of 0 and ∞ in $\mathcal{C} \times \mathcal{D}$ is the subvariety, V', supporting the divisors (f) and (g). Remove V' from $\mathcal{C} \times \mathcal{D}$; the open set $U \subset \mathcal{C} \times \mathcal{D}$ that remains dominates $\mathcal{W}$ via the morphism $\beta^{-1}|_U$.

Since the support of the divisors (f) and (g) consists of irreducible components of the form $\mathcal{C} \times y$ and $x \times \mathcal{D}$, the open set U is a product of open curves, which we denote $C_1 \times C_2$.

The restriction of ρ to a map $\mathcal{W}_d \to \mathcal{W}$ is an étale covering, and $C_1 \times C_2$ dominates $\mathcal{W}$ via a morphism. Theorem 2.1 applies, and the surface $\mathcal{S}_d$ is DPC. $\square$

2.6. With Theorem 2.2 we prove Birch and Swinnerton-Dyer for a large class of varieties.

Theorem 2.3. *Let k denote the finite field $\mathbb{F}_q$, q a prime power, and let $\mathcal{S}$ denote the smooth DPCT surface over $\mathbb{P}^1_k$ constructed as above. Then the conjecture of Birch and Swinnerton-Dyer is true for the Jacobian of the generic fiber of $\mathcal{S}_d$, an abelian variety over $k(t^{1/d})$.*

Proof. This is a corollary to Theorem 2.2. It follows immediately from the DPCT property of $\mathcal{S}$ and the equivalence of the Tate conjecture for $\mathcal{S}_d$ to BSD for the Jacobian of its generic fiber. $\square$

3. The generic fiber of a fibered surface

3.1. We apply Theorem 2.2 to provide curves of infinitely many genera whose Jacobians satisfy the conjecture of Birch and Swinnerton-Dyer. Assume in what follows that k is a perfect field, and define $f(x_0, x_1) := \frac{\prod_i (x_0-\kappa_i x_1)^{m_i}}{\prod_k (x_0-\lambda_k x_1)^{r_k}}$ and $g(y_0, y_1) := \frac{\prod_e (y_0-\nu_e y_1)^{s_e}}{\prod_j (y_0-\mu_j y_1)^{n_j}}$, rational functions on $\mathbb{P}^1_k$.

Let $K = k(t)$ and consider the bidegree (m, n) curve C' in $\mathbb{P}^1_K \times \mathbb{P}^1_K$ defined by the vanishing of:

$$G := t\prod_i (x_0 - \kappa_i x_1)^{m_i} \prod_j (y_0 - \mu_j y_1)^{n_j} - \prod_k (x_0 - \lambda_k x_1)^{r_k} \prod_e (y_0 - \nu_e y_1)^{s_e}, \tag{1}$$

with all κ_i, μ_j, λ_k, and $\nu_e \in k$, and with κ_i, λ_k all distinct and μ_j, ν_e all distinct.

This model corresponds to the generic fiber of the surface S' described in the previous section, in the particular case where $\mathcal{C} = \mathcal{D} = \mathbb{P}^1$, and $f \in k(\mathcal{C})$ and $g \in k(\mathcal{D})$ are defined as above. We determine sufficient conditions for geometric irreducibility, resolve any singularities, and determine the geometric genus of a smooth, projective model.

3.2.

Theorem 3.1. *Suppose that the exponents in the polynomial G are relatively prime. Then*:

1. *The curve C' is absolutely irreducible.*
2. *If the exponents are also prime to* $\operatorname{char}(K)$*, then the unique smooth projective model C has geometric genus*

$$g = (m-1)(n-1) - \sum_{(i,h)} \delta(m_i, p_h) - \sum_{(e,j)} \delta(n_e, k_j),$$

where $\delta(a, b) = \frac{(a-1)(b-1)}{2} + \frac{((a,b)-1)}{2}$, and the sums are taken over all pairs (i, h), (e, j).

We first prove the irreducibility result, which relies on two attributes intrinsic to the equation defining our curve. First, by restricting f to $\mathcal{C}$ and g to $\mathcal{D}$ we require that the divisors over 0 and ∞ on the fibered surface $\mathcal{S}' \to \mathbb{P}^1$ are vertical and horizontal, a geometric restriction that imposes a separation of variables on the defining equation G, and we use this to show that the rational functions tf and g admit compatible decompositions. Second, we exploit the fact that, while g is defined over k, tf must be defined over the extension K.

3.3. In the case where the degree of tf or g is 1, irreducibility is obvious. We assume in what follows that $\deg(f), \deg(g) > 1$. The following lemma is due to Fried [Fri73]; we prove a more general statement.

Lemma 3.1. *Let L/F and K/F denote finite, separable field extensions, and assume that $K \otimes_F L$ is not a field. Then there exist intermediate fields $F \subset K' \subset K$ and $F \subset L' \subset L$ with K', $L' \neq F$, such that the Galois closures ${K'}^{\mathrm{gal}}$ and ${L'}^{\mathrm{gal}}$ are isomorphic.*

Proof. Embed K and L in an algebraic closure $\bar{F}$ of F. Let $q(x)$ and $p(x)$ denote irreducible polynomials over F with $L = F[x]/(q)$ and $K = F[x]/(p)$. Since we assume that $L \otimes_F K \simeq K[x]/(q)$ is not a field, it follows that (q) is not maximal, and q factors in $K[x]$. Let K_1 denote the field generated by the coefficients of the irreducible factors of q over K. This is a proper extension of F, and we let K_1^{gal} denote its Galois closure.

$$\begin{tikzcd}
L^{\mathrm{gal}} & & K_1^{\mathrm{gal}} \arrow[ll] \\
L \arrow[u] & & K_1 \arrow[u] \arrow[llu] \\
& F \arrow[lu] \arrow[ru] &
\end{tikzcd}$$

The irreducible factors of $q(x)$ over K correspond to the orbits of its roots by $\mathrm{Gal}(KL^{\mathrm{gal}}/K)$. The irreducible factors of $q(x)$ over K_1 correspond to the orbits of its roots by $\mathrm{Gal}(K_1L^{\mathrm{gal}}/K_1) = \mathrm{Gal}(L^{\mathrm{gal}}/K_1)$. Since these groups are isomorphic, and since $q(x)$ factors over K, it follows that $q(x)$ factors over K_1. This implies that $L \otimes_F K_1$ is not a field, and we repeat this construction. Factor p in $L[x]$, and let L_1 denote the proper extension of F generated by the coefficients of the factors of p over L. Continuing in this manner we "replace" L and K with intermediate fields lying between L and F and between K and F. This process of constructing intermediate fields within finite extensions must terminate, and at the final stage we have $L_i \subset K_j^{\mathrm{gal}}$ and $K_j \subset L_i^{\mathrm{gal}}$.

$$\begin{tikzcd}
L_i^{\mathrm{gal}} & & K_j^{\mathrm{gal}} \\
L_i \arrow[u] \arrow[rru] & & K_j \arrow[u] \arrow[llu] \\
& F \arrow[lu] \arrow[ru] &
\end{tikzcd}$$

These containments, combined with the minimality of Galois closures, proves the isomorphism $L_i^{\mathrm{gal}} \simeq K_j^{\mathrm{gal}}$. $\square$

3.4. We combine this result with a lemma that shows that a k-morphism $\mathbb{P}^1 \to \mathbb{P}^1$ that factors over $k(t)$ must have a compatible factorization over k.

Lemma 3.2. *Let $k = \bar{k}$ and let K/k denote a field extension. Let $g : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a morphism and denote also by g the morphism $\mathbb{P}^1_K \to \mathbb{P}^1_K$ induced by the base change to K. Suppose that g factors over K as*

$$g : \mathbb{P}^1_K \xrightarrow{g_2} \mathbb{P}^1_K \xrightarrow{g_1} \mathbb{P}^1_K.$$

Then this factorization may be defined over k. More precisely, there is a factorization:

$$g : \mathbb{P}^1_k \xrightarrow{g_2} \mathbb{P}^1_k \xrightarrow{g_1} \mathbb{P}^1_k$$

whose base change to K is the assumed K-factorization.

Proof. Let $k(z)$ denote the function field of $\mathbb{P}^1_k$, with $k(y)/k(z)$ the finite, separable, algebraic extension corresponding to the morphism $g : \mathbb{P}^1_k \to \mathbb{P}^1_k$. Fix an algebraically closed field Ω containing $K(y)$, let ℓ denote the Galois closure of $k(y)/k(z)$ in Ω, and note that $L := \ell K(z) = \ell K(y)$ and $\ell \cap K(z) = k(z)$. Hence $L/K(z)$ is Galois with group $G \simeq \mathrm{Gal}(\ell/k(z))$. Analogously $\mathrm{Gal}(\ell/k(y)) \simeq \mathrm{Gal}(L/K(y))$.

$$\begin{tikzcd}
\ell \arrow[r] & L = \ell K(y) \\
k(y) \arrow[u] \arrow[r] & K(y) \arrow[u] \\
k(z) \arrow[u] \arrow[r] & K(z) \arrow[u]
\end{tikzcd}$$

Now denote by F_K the extension of $K(z)$ corresponding to the morphism $g_1 : \mathbb{P}^1_K \to \mathbb{P}^1_K$. Let H denote the subgroup of $G = \mathrm{Gal}(L/K(z))$ with fixed field F_K. Let F_k denote the intersection $\ell \cap F_K$, as in the diagram below. It is clear that $K F_k = F_K$.

$$\begin{tikzcd}
k(y) \arrow[r] & K(y) \\
F_k \arrow[u] \arrow[r] & F_K \arrow[u, "g_2^*"'] \\
k(z) \arrow[u] \arrow[r] & K(z) \arrow[u, "g_1^*"']
\end{tikzcd}$$

Indeed, since ℓ/F_k is a Galois extension, and since $\ell F_K = \ell K(y)$, we have $\mathrm{Gal}(\ell/F_k) = \mathrm{Gal}(L/F_K)$. Thus $[\ell : F_k] = [L : F_K]$. It follows from this equality, and the fact that $KF_k \subseteq F_K$, that $KF_k = F_K$. The inclusion $F_k \hookrightarrow F_K$ corresponds to the base change $\mathbb{P}^1_K \to \mathbb{P}^1_k$, and the desired k-factorization of g is the morphism of curves corresponding to the inclusions $k(z) \hookrightarrow F_k \hookrightarrow k(y)$. □

3.5.

Lemma 3.3. *Let k denote an algebraically closed field contained in K, and let h denote a morphism $\mathbb{P}^1_K \to \mathbb{P}^1_K$. Suppose that there exist three k-rational points P_i with $h^{-1}(P_i)$ consisting of k-rational points. Then h is defined over k.*

Proof. By composing with a k-morphism, we may assume that the points P_i are 0, 1, and ∞. Write $h := \gamma \frac{\prod(x-a_i)}{\prod(x-b_i)}$. Since $h^{-1}(0)$ consists of k-rational points, we have $a_i \in k$. The b_i similarly lie in k, since $h^{-1}(\infty)$ consists of k-points. It follows immediately that $\gamma \in k$. □

With notation as above, we can now prove the proposition:

Proposition 3.1. *If the exponents m_i, n_e, k_j, p_h have no common divisor then the curve C' is absolutely irreducible.*

Proof. We construct our curve C' above as the fiber product of the morphisms $\mathbb{P}^1_K \to \mathbb{P}^1_K$ defined by the rational functions tf and g and assume that C' is reducible.

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & \mathbb{P}^1_x \arrow[d, "tf"] \\ \mathbb{P}^1_y \arrow[r, "g"'] & \mathbb{P}^1_z \end{tikzcd}$$

Let $K(x)$ denote the function field of $\mathbb{P}^1_x$; this is an algebraic extension of $K(z)$, the function field of the curve $\mathbb{P}^1_z$ below. Similarly for $K(y)$. Embed $K(y)$ and $K(x)$ in an algebraic closure $\overline{K(z)}$ of $K(z)$. Since C' is reducible the total ring of fractions, $K(x) \otimes_{K(z)} K(y)$ is not a field, and Lemma 3.1 applies. The intermediate field extensions implied by Lemma 3.1 correspond to a factorization over K of the morphisms tf and g. So we find f_1, f_2, g_1, and g_2 with $tf(x) = f_1(f_2(x))$, $g(y) = g_1(g_2(y))$, so that the splitting fields of $f_1(x) - z$ and $g_1(y) - z$ over $K(z)$ are isomorphic.

By Lemma 3.2, the factorization $g = g_1(g_2(y))$ may be defined over $\bar{k}$.

This, combined with the fact that $g_1(y) - z$ and $f_1(x) - z$ have isomorphic splitting fields over $K(z)$, yields the field extensions in the diagram below.

$$\begin{tikzcd}
M & m \arrow[l] \\
K(x_1) \arrow[u] & \\
K(z) \arrow[u, "f_1^*"] & k(z) \arrow[l] \arrow[uu]
\end{tikzcd}$$

The inclusion $k(z) \hookrightarrow m$ corresponds to a morphism $\mathbb{P}^1_k \to \mathbb{P}^1_k$. Over K this morphism factors through f_1. We again apply Lemma 3.2, and it follows that f_1 may be defined over $\bar{k}$.

Consider now the following decompositions of the morphism tf over K:

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[r, "f_2"] \arrow[d, "f"] & \mathbb{P}^1 \arrow[d, "f_1"] \\
\mathbb{P}^1 \arrow[r, "t"] & \mathbb{P}^1.
\end{tikzcd}$$

Let P_1 and P_2 denote two $\bar{k}$-rational points with $f_1(P_1) = 0$ and $f_1(P_2) = \infty$. Suppose we have $P \neq P_1$ with $f_1(P) = 0$. We have $\operatorname{div}(tf) = \operatorname{div}(f)$ and, since f is defined over k, the points in $\operatorname{supp}\operatorname{div}(f)$ have coordinates in $\bar{k}$. It follows that $f_2^{-1}(P)$, $f_2^{-1}(P_1)$, and $f_2^{-1}(P_2)$ are all defined over $\bar{k}$. Lemma 3.3 then implies that the function f_2 is defined over $\bar{k}$, a contradiction, since $tf = f_1 \circ f_2$ and f_1 is defined over $\bar{k}$. It follows that f_1 is fully ramified over 0. Similarly, f_1 is fully ramified over ∞, and the Riemann–Hurwitz formula implies that there are no other branch points. Thus f_1 is of the form $\gamma \frac{(x-a)^d}{(x-b)^d}$, and the exponents in our defining equation have a common factor.

This proves the proposition, and the first half of Theorem 3.1. □

3.6. We next restrict our construction to rational functions f and g chosen so that the exponents defining G are relatively prime, and we determine a closed formula for the geometric genus of the smooth, projective model of the curve in $\mathbb{P}^1_K \times \mathbb{P}^1_K$ defined by the vanishing of G, in terms of these exponents.

In Section 3.7 we recall some definitions and results on singularities of plane curves and refer the reader to [Ser88] for further details. In Section 3.8 we apply these results to determine a closed formula for the genera of our smooth curves.

3.7. Let C' denote an irreducible algebraic curve with normalization $\pi : C \dashrightarrow C'$. Let $\mathcal{O}'_Q$ denote the local ring at the singular point Q on C', and $\mathcal{O}_Q$ its normalization. The geometric genus of C may be computed in terms of the arithmetic genus of C' and a numerical invariant, δ, determined by its singular points.

Definition 3.1.

$$\delta_Q = \dim_K \mathcal{O}_Q/\mathcal{O}'_Q,$$

and

$$\delta = \sum_Q \delta_Q.$$

Lemma 3.4. *Let $p_a(C')$ denote the arithmetic genus of an irreducible curve C'. Then, with notation as above*

$$p_a(C) = p_a(C') - \delta.$$

Proof. Let $\pi_*\mathcal{O}_C$ denote the direct image sheaf under the normalization map π. We have the exact sequence

$$0 \to \mathcal{O}_{C'} \hookrightarrow \pi_*\mathcal{O}_C \twoheadrightarrow \pi_*\mathcal{O}_C/\mathcal{O}_{C'} \to 0.$$

Taking Euler characteristics yields

$$\chi(C', \mathcal{O}_{C'}) = \chi(C', \pi_*\mathcal{O}_C) + \chi(C', \pi_*\mathcal{O}_C/\mathcal{O}_{C'}).$$

By the definition of arithmetic genus, this is just

$$1 - p_a(C') = 1 - p_a(C) + \chi(C', \pi_*\mathcal{O}_C/\mathcal{O}_{C'}),$$

so that

$$p_a(C') = p_a(C) - \chi(C', \pi_*\mathcal{O}_C/\mathcal{O}_{C'}).$$

We compute $\chi(C', \pi_*\mathcal{O}_C/\mathcal{O}_{C'})$ using Čech cohomology, noting that the sheaf $\pi_*\mathcal{O}_C/\mathcal{O}_{C'}$ is supported on the set S of singular points of C', and the result follows. □

Any point P mapping to Q at any stage of the resolution $C \dashrightarrow C'$, including Q, is called a nearby point, and δ_Q depends only on the multiplicities of the curve at Q and at each of the other nearby points P mapping to Q.

Proposition 3.2. *For each strict transform C_i of C' let $P_{i,1}, \ldots, P_{i,n_i}$ denote the points nearby to Q, and let $m_{P_{i,j}}$ denote the multiplicity of C_i at $P_{i,j}$. Then*

$$\delta_Q = \sum_{i,j} \frac{m_{P_{i,j}}(m_{P_{i,j}} - 1)}{2}.$$

Proof. That $\dim_K \mathcal{O}_Q/\mathcal{O}'_Q = \frac{m_Q(m_Q-1)}{2}$ for a single blow-up is detailed in [Ful04].

The inclusion $\mathcal{O}''_Q \subseteq \mathcal{O}'_Q \subseteq \mathcal{O}_Q$ of semi-local rings for successive transforms give us the exact sequence

$$0 \to \mathcal{O}'_Q/\mathcal{O}''_Q \hookrightarrow \mathcal{O}_Q/\mathcal{O}''_Q \twoheadrightarrow \mathcal{O}_Q/\mathcal{O}'_Q \to 0,$$

and the proposition then follows from the additivity of vector space dimensions:

$$\dim_K \mathcal{O}_Q/\mathcal{O}''_Q = \dim_K \mathcal{O}_Q/\mathcal{O}'_Q + \dim_K \mathcal{O}'_Q/\mathcal{O}''_Q. \qquad \square$$

3.8. We determine the singular locus of C and show that each singular point is analytically equivalent to one defined by $x^\alpha = y^\beta$.

Lemma 3.5. *Let C' denote the curve over $K = k(t)$ defined by the vanishing of G, and let κ_i, μ_j, λ_k, and ν_e be as in* (1). *Then the morphism $C' \to \operatorname{Spec} K$ is smooth, except where $x_0 = \kappa_i x_1$ and $y_0 = \nu_e y_1$ with exponents m_i, s_e greater than one, and where $x_0 = \lambda_k x_1$ and $y_0 = \mu_j y_1$ with exponents r_k, n_j greater than one.*

Proof. It is clear by the Jacobian criterion that C' is singular where claimed, and we show that there are no other singular points. It is sufficient to consider the affine model $G(x, y)$, with $x = \frac{x_0}{x_1}$ and $y = \frac{y_0}{y_1}$. Set $f_1(x) := \prod_i (x - \kappa_i)^{m_i}$, $f_2(x) = \prod_k (x - \lambda_k)^{r_k}$, $g_1(y) = \prod_j (y - \mu_j)^{n_j}$, and $g_2(y) := \prod_e (y - \nu_e)^{s_e}$.

The condition that $\partial G/\partial x$ vanish gives

$$t f_1'(x) g_1(y) - f_2'(x) g_2(y) = 0.$$

Then, assuming $f_1(x), g_2(y) \neq 0$ we obtain:

$$f_1'(x) f_2(x) - f_1(x) f_2'(x) = 0.$$

Since f is defined over k, any solution lies in $\bar{k}$. A similar analysis, assuming $f_2(x), g_1(y) \neq 0$, shows that any solution to $\partial G/\partial y = 0$ must also lie in $\bar{k}$.

The proposition follows immediately from the fact that for $(x, y) \in \bar{k}^2$ we have $f_2(x) g_2(y) \in \bar{k}$ and $t f_1(x) g_1(y) \notin \bar{k}$, unless $f_1(x) g_1(y) = 0$. $\square$

Lemma 3.6. *Consider the singularity at $Q = (\kappa_p, \nu_q)$ on the affine curve C' defined by the equation*

$$t \prod_i (x - \kappa_i)^{m_i} \prod_j (y - \mu_j)^{n_j} - \prod_k (x - \lambda_k)^{r_k} \prod_e (y - \nu_e)^{s_e} = 0,$$

with exponents prime to $\operatorname{char}(K)$. *The invariant δ_Q is equal to that of the singularity at the origin defined by the affine equation $x^{m_p} - y^{s_q} = 0$.*

Proof. We first note that δ is an analytic invariant of the singularity, i.e., that

$$\dim_K \mathcal{O}_Q/\mathcal{O}'_Q = \dim_K \hat{\mathcal{O}}_Q/\hat{\mathcal{O}}'_Q.$$

Let $\mathfrak{m}'$ denote the maximal ideal at Q and $\mathfrak{m}$ its image in the normalization. Since $\mathcal{O}_Q/\mathcal{O}'_Q$ is finite-dimensional, there exist integers r, and r' such that

$$\mathcal{O}_Q/\mathcal{O}'_Q \simeq \big(\mathcal{O}_Q/\mathfrak{m}^N\big)/\big(\mathcal{O}'_Q/\mathfrak{m}'^N\big) \quad \text{for } N \geqslant r,$$

and

$$\hat{\mathcal{O}}_Q/\hat{\mathcal{O}}'_Q \simeq \big(\hat{\mathcal{O}}_Q/\mathfrak{m}^N\big)/\big(\hat{\mathcal{O}}'_Q/\mathfrak{m}'^N\big) \quad \text{for } N \geqslant r'.$$

The result then follows from the fact that, for any N,

$$\big(\mathcal{O}_Q/\mathfrak{m}^N\big)/\big(\mathcal{O}'_Q/\mathfrak{m}'^N\big) \simeq \big(\hat{\mathcal{O}}_Q/\mathfrak{m}^N\big)/\big(\hat{\mathcal{O}}'_Q/\mathfrak{m}'^N\big).$$

Indeed, for any N, we have $(\mathcal{O}_Q/\mathfrak{m}^N) \simeq (\hat{\mathcal{O}}_Q/\mathfrak{m}^N)$.

It is left to show the analytic equivalence of our singularities to those defined by $x^{m_q} - y^{s_q} = 0$.

We change variables to move the singularity to the origin, so an affine model of C' is defined by

$$x^{m_p} \prod_{i \neq p} (x - \kappa_i)^{m_i} \prod_j (y - \mu_j)^{n_j} - y^{s_q} \prod_k (x - \lambda_k)^{r_k} \prod_{e \neq q} (y - \nu_e)^{s_e} = 0.$$

The image of the complete local ring $k[\![x, y]\!]/(x^{m_p} - y^{s_q})$ in its normalization is isomorphic to that of

$$k[\![x, y]\!]\Big/\bigg(x^{m_p} \prod_{i \neq p} (x - \kappa_i)^{m_i} \prod_j (y - \mu_j)^{n_j} - y^{s_q} \prod_k (x - \lambda_k)^{r_k} \prod_{e \neq q} (y - \nu_e)^{s_e}\bigg).$$

Indeed, by Hensel's lemma, for every non-zero α and β, the elements $(x - \alpha)$ and $(y - \beta)$ have nth roots for every n prime to $\operatorname{char}(K)$.

Then the isomorphism is given by $x \mapsto x(\prod_{i \neq p} (x - \kappa_i)^{-m_i} \prod_j (y - \mu_j)^{-n_j})^{1/m_p}$ and $y \mapsto y(\prod_k (x - \lambda_k)^{-r_k} \prod_{e \neq q} (y - \nu_e)^{-s_e})^{1/s_q}$. $\square$

Consider now a singular point in the plane defined by the affine equation $x^{\alpha n} = y^{\beta n}$, with $(\beta, \alpha) = 1$. We have the following closed formula:

Proposition 3.3.

$$\delta_Q = \frac{(\beta n - 1)(\alpha n - 1) + (n - 1)}{2}.$$

Proof. Changing coordinates if necessary, we assume that $\alpha \leqslant \beta$, and we proceed by induction. When $\alpha = 1$ the singularity is resolved in β blow-ups; the multiplicity of the curve at the singular point is n at each stage; and $\delta = \frac{\beta n(n-1)}{2}$. For $\alpha > 1$ we assume $\alpha < \beta$, since $(\alpha, \beta) = 1$.

Assume for some α_k that the proposition holds for every α with $\alpha < \alpha_k < \beta$, and consider a singularity defined by $x^{\alpha_k n} = y^{\beta n}$. Since $\beta > \alpha_k$ we may write $\beta = q\alpha_k + r$, $r < \alpha_k$. There are q nearby points with multiplicities $\alpha_k n$, giving a contribution to δ of $\frac{q\alpha_k n(\alpha_k n-1)}{2}$, leaving the singularity in the form $x^{\alpha_k n} - y^{(\beta - q\alpha_k)n}$, with $\beta - q\alpha_k < \alpha_k$. Applying the hypothesis yields

$$\delta = \frac{q\alpha_k n(\alpha_k n - 1)}{2} + \frac{(\alpha_k n - 1)((\beta - q\alpha_k)n - 1) + (n - 1)}{2}$$
$$= \frac{(\beta n - 1)(\alpha_k n - 1) + (n - 1)}{2},$$

and the proposition follows by induction. □

We recall that the arithmetic genus of an (m,n) curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is $(m-1)(n-1)$, an immediate result of the adjunction formula, and our genus result follows. Combined with the irreducibility proposition, this completes the proof of the theorem.

4. Large ranks

4.1. Combining the results of Sections 2 and 3 we may construct towers of DPCT surfaces and determine the genera of the generic fibers of these surfaces. When $k = \mathbb{F}_q(t)$ the Jacobian of the generic fiber of each surface in the tower satisfies BSD. Applying known results on analytic ranks we obtain a parameterized family of elliptic curves that obtain unbounded ranks in towers.

Theorem 4.1. *Let $p \geqslant 5$, q a power of p. For $a, b \in \mathbb{F}_q$, $a, b \neq 0, 1$, let $E_{a,b}/\mathbb{F}_q(t)$ denote the family of curves defined by*

$$y^2 + (b - at)xy - \left(at^2 - bt\right)y = x^3 + (a + b + 1)tx^2 + \left[(a + 1)b + a\right]t^2x + abt^3.$$

Then:

(1) *The curves $E_{a,b}$ are elliptic curves, and they satisfy the conjecture of Birch and Swinnerton-Dyer over the fields $\mathbb{F}_q(t^{1/d})$, where $(d, p) = 1$.*
(2) *For $a \neq 2$ the elliptic curves $E_{a,a}$ obtain unbounded rank over $\mathbb{F}_q(t^{1/d})$ as d runs through integers prime to p.*

In Section 4.3, example (7), we show that the curves $E_{a,b}$ satisfy BSD, which will prove part (1) of Theorem 4.1. We complete the proof in Section 4.4, example (3), where we show that the curves $E_{a,a}$ obtain unbounded analytic ranks in towers.

4.2. Analytic ranks

We produce the curves $E_{a,b}$ via the simplest case of our DPCT construction. Let $\mathcal{C} = \mathcal{D} = \mathbb{P}^1$, and let f and g denote degree 2 rational functions in $K(\mathbb{P}^1)$. Resolving the rational map $\frac{f}{g} : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ we obtain a fibered surface, possibly singular, with generic fiber defined by Eq. (1).

We showed that the choice of exponents defining our bidegree (m,n) pencils determines the genus of the generic fiber of a smooth model, and in our current set-up $m = n = 2$. Hence, if the generic fiber is smooth, and with a rational point, then the $(2,2)$ curve defined by Eq. (1) is an elliptic curve.

Ulmer [Ulm07] proves that certain L-series vanish to high order at $s = 1$ in towers of function fields.

Theorem 4.2. *(See [Ulm07].) Let $K = \mathbb{F}_q(t^{1/d})$ and let E/K denote a non-isotrivial elliptic curve. Let $\mathfrak{n}'$ denote the part of the conductor of E/K prime to 0 and ∞. Suppose that the degree of $\mathfrak{n}'$ is odd and that $p \geqslant 5$.*

Then, for $d = q^n + 1$,

$$\operatorname{ord}_{s=1} L(E/K, s) \geqslant \frac{d}{2n} - c,$$

where c is a constant independent of n.

In Section 4.3 we study each family of $(2,2)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$ constructed as above. In Section 4.4 we determine which of the elliptic curves that arise satisfy the hypotheses of Theorem 4.2.

4.3. *BSD for our* $(2,2)$ *curves*

With notation as in Section 3, we denote by the partitions $[m_i][r_k][n_j][s_e]$ the family of $(2,2)$ curves defined by G with corresponding exponents.

(1) [2][2][2][2]
The curve is reducible, and we omit further discussion.

(2) [2][2][2][1, 1]
Any curve in this family is singular, and a smooth model is birational to $\mathbb{P}^1$.

(3) [2][1, 1][1, 1][2]
Any model is singular, and a smooth model is birational to $\mathbb{P}^1$.

(4) [2][2][1, 1][1, 1]
This is the first example of a smooth family. This family may be defined by the equation

$$tx_0^2 y_0(y_0 - y_1) + x_1^2 y_1(y_0 - ay_1) = 0.$$

The equation has Weierstrass form

$$y^2 = x^3 + Ax + B,$$

with

$$A = -432t^2\bigl(a^2 - a + 1\bigr),$$
$$B = -1728t^3\bigl(2a^3 - 3a^2 - 3a + 2\bigr),$$

and the curve is isotrivial for any a.

(5) [1, 1][2][1, 1][2]
We have the pencil defined by $tx_0y_0x_1y_1 - (x_0 - x_1)^2(y_0 - y_1)^2 = 0$.
The generic fiber has Weierstrass equation

$$y^2 = x^3 + \bigl(t^2 + 8t\bigr)x^2 + 16t^2x$$

and j-invariant

$$j = \frac{(t^2+16t+16)^3}{t(t+16)}.$$

This is our first example of a non-isotrivial elliptic curve that satisfies BSD by the results of Sections 2 and 3.

(6) [1, 1][1, 1][1, 1][2]
We consider the family of genus one curves whose model in $\mathbb{P}^1 \times \mathbb{P}^1$ is given by the vanishing of the polynomial

$$tx_0(x_0-x_1)y_0(y_0-y_1) - x_1y_1^2(x_0-ax_1).$$

This has Weierstrass form:

$$y^2 = x^3 + Ax + B$$

with

$$A = -27\bigl(t^4 + 8(2a-1)t^3 + 16\bigl(a^2-a+1\bigr)t^2\bigr)$$

and

$$B = 54\bigl(t^6 + 12(2a-1)t^5 + 24\bigl(5a^2-5a+2\bigr)t^4 - 32\bigl(2a^3-3a^2-3a+2\bigr)t^3\bigr).$$

The j-invariant is

$$\begin{aligned}
&\bigl[t^6 + (-24+48a)t^5 + \bigl(240-816a+816a^2\bigr)t^4\\
&\quad + \bigl(-1280+5376a-8448a^2+5632a^3\bigr)t^3\\
&\quad + \bigl(3840-16896a+29952a^2-26112a^3+13056a^4\bigr)t^2\\
&\quad + \bigl(-6144+24576a-43008a^2+49152a^3-30720a^4+12288a^5\bigr)t\\
&\quad + \bigl(4096-12288a+24576a^2-28672a^3+24576a^4-12288a^5+4096a^6\bigr)\bigr]\\
&\quad /\bigl[a^2(a-1)^2\bigl(t^2+8(2a-1)t+16\bigr)\bigr].
\end{aligned}$$

This family of curves satisfies BSD.

(7) [1, 1][1, 1][1, 1][1, 1]
We consider the 2-parameter family of curves in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the equation

$$tx_0(x_0-ax_1)y_1(y_0-y_1) - x_1(x_0-x_1)y_0(y_0-by_1) = 0,$$

with $a, b \neq 0, 1$.
An equation in Weierstrass form is:

$$E_{a,b}\colon y^2 + (b-at)xy - \bigl(at^2-bt\bigr)y = x^3 + (a+b+1)tx^2 + \bigl[(a+1)b+a\bigr]t^2x + abt^3.$$

The fibered surface defined by this equation is smooth, with generic fiber an elliptic curve. By the results of Section 2 the surface is DPCT. It follows that every member of the 2-parameter family $E_{a,b}$ defined by the Weierstrass equation above satisfies BSD over the fields $\mathbb{F}_q(t^{1/d})$. In the next section we prove the second half of the theorem.

4.4. Large ranks

In this section we find the reduction types of our $(2,2)$ curves and determine where Theorem 4.2 may be applied to establish large ranks of our curves.

(1) $[1,1][2][1,1][2]$
We computed above the j-invariant:

$$j = \frac{(t^2+16t+16)^3}{t(t+16)}.$$

Away from 0 and ∞ the curve has bad reduction only at $t+16$, and the type of reduction is multiplicative. So $\mathfrak{n}'$ has odd degree, and $E/\mathbb{F}_q(t^{1/d})$ obtains arbitrarily large rank as $d \to \infty$.

(2) $[1,1][1,1][1,1][2]$
We write the curve in the Weierstrass form

$$y^2 + txy - at^2y = x^3 - (a+1)x^2 + at^2x$$

and compute its discriminant

$$\Delta = a^2t^6(a-1)^2\big(t^2 + 8(2a-1)t + 16\big).$$

The elliptic curve has multiplicative reduction at both places dividing

$$t^2 + 8(2a-1)t + 16,$$

and Theorem 4.2 does not apply.

(3) $[1,1][1,1][1,1][1,1]$
We set $a = b$ in this 2 parameter family, and an affine model is given by

$$tx(x-a)(y-1) - y(x-1)(y-a),$$

with $a \neq 0, 1$.
A Weierstrass equation is

$$E_{a,a}\colon y^2 + (a-at)xy - \big(at^2 - at\big)y = x^3 + (2a+1)tx^2 + \big[a^2+2a\big]t^2x + a^2t^3.$$

The discriminant is

$$\Delta = a^2(a-1)^4t^4(t-1)^2\big(a^2t^2 - 2\big(a^2-8a+8\big)t + a^2\big),$$

and the curve has multiplicative reduction at the place $t-1$.

The quadratic $(a^2t^2 - 2(a^2 - 8a + 8)t + a^2)$ is prime to c_4 and, for $a \neq 2$, the curve has multiplicative reduction at both places dividing the quadratic.

We again apply [Ulm07], and the members of the 1-parameter family of elliptic curves $E_{(a,a)}/\mathbb{F}_q(t^{1/d})$ obtain arbitrarily large rank as $d \to \infty$, which completes the proof of our theorem.

4.5. Bidegree (m,n) curves

This paper describes a method of constructing curves over $\mathbb{F}_q(t)$, of infinitely many genera, whose Jacobians satisfy BSD. Exploiting only the simplest case, we produced a one-parameter family of elliptic curves that obtain large rank in towers of function field extensions. A further step is to consider curves of bidegree (m,n) that arise in our construction and determine whether other families of elliptic curves of large rank may be obtained. We believe that curves of arbitrary genus may be constructed using our methods, producing abelian varieties of any dimension that satisfy BSD. By a further analysis of the conductors of the Galois representations that arise, we expect to apply results from [Ulm07] to show that some of these varieties obtain large rank.

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